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The global phase diagram of a modular invariant two dimensional statistical model [1]

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Abstract A generalization of the Coulomb Gas model with modular $SL(2,Z)$-symmetry allows for a discrete infinity of phases which are characterized by the condensation of dyonic pseudoparticles and the breaking of parity and time reversal. Here we study the phase diagram of such a model by using renormalization group techniques. Then the symmetry $SL(2,Z)$ acting on the two-dimensional parameter space gives us a nested shape of its global phase diagram and all the infrared stable fixed points. Finally we propose a connection with the 2-dimensional Conformal Field Theory description of the Fractional Quantum Hall Effect.

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1 Introduction.

In 1982 Cardy and Rabinovici, following an original suggestion by G. $'$t Hooft [1], have built a simple model derived by means of dimensional reduction of an Abelian Z_p lattice gauge theory in four dimensions [2, 3]. The relevant parameters of the theory are the coupling strength g and the θ-angle. By integrating out the gauge fields and by using the electric and magnetic current densities as dynamical variables, the partition function of such a model is mapped onto the one of a "generalized" or "modular invariant Coulomb Gas" (**mCG**). The model does preserve both parity (**P**) and time-reversal (**T**) symmetries and so does any of its phases, given by condensates of particles carrying both electric and magnetic charge (**dyons**). Thus charges of both sign appear in the field theory associated to each infrared (IR) fixed point.

Moreover all the phases are subjected to the "charge neutrality" constraint, that is, the sum of the total electric and magnetic charge carried by the condensed particles must be zero. This implies that, in each phase, the number of "particles" must be equal to the number of "antiparticles", as usually happens within the condensate phases of a CG [4].

In this paper we shall use a modified version of the mCG where an uniform background charge is introduced [5]. In order to derive the phase diagram we will perform the Renormalization Group (**RG**) analysis of the modified mCG. The relevant $SL(2, Z)$ (modular) symmetry analyzed in [2, 3] is still a symmetry of the partition function, provided one lets the transformations act onto the uniform background as well. That imposes strong constraints onto the global phase diagram, which acquires the characteristic nested shape [6, 7], thus strongly suggesting an unified picture of the various phases, as well as of the transitions between phases. Then we are able to identify a discrete infinite set of IR stable fixed points on the θ-axis at $\theta/2\pi$ given by any rational number. Each of them controls the IR physics of a region in the plane of the parameters $(\frac{\theta}{2\pi}, \frac{1}{g})$ and corresponds to a condensate of dyons carrying both electric and magnetic charge. The background charge and the particle density do uniquely fix the charges of the condensate.

The field theory that is supposed to describe the system close to an IR fixed point breaks parity(**P**) and charge conjugation(**C**) (now the charges of the dyons have all the same sign). It can be identified with a 2-dimensional(**2D**) Conformal Field Theory (**CFT**) with central charge $c = 1$, described by a chiral scalar field compactified on a circle of radius $R^2 = q_e/q_m$, where q_e and q_m are the electric and the magnetic charges of the dyons [8]. That is a nontrivial result which, together

with the properties of the phase diagram, suggests the relevance of the model for the 2D CFT description of the Fractional Quantum Hall Effect (**FQHE**).

The paper is organized as follows:

- In section 2 we introduce the dimensionally reduced CR model. Then we briefly sketch its gauge formulation and focus on its equivalent description as a $SL(2,Z)$ (modular) invariant mCG, which is the one used for the RG analysis;
- In section 3 we use a first-order RG calculation in order to derive the physical properties of the nonneutral phases, identified by the condensation of dyons ;
- In section 4 we perform nonlinear RG calculations, which give us the flow in the two parameters space ($\frac{\theta}{2\pi},\frac{1}{g}$),i.e. a complex plane;
- In section 5 we get the global phase diagram by applying the RG results and the infinite discrete $SL(2,Z)$-symmetry, which acts on the complex plane, introduced in sec. 5. This enables us to find all the RG stable fixed points and the relative domains, where each condensate is relevant;
- In section 6 we conjecture that the effective description at the IR fixed points of our mCG is provided by a 2D chiral CFT. We briefly explore the consequences of such an identification, particularly for what concerns its relevance for the FQHE;
- In section 7 we provide the main conclusions and discuss open problems.

2 The CR model with a background charge density.

In this section we shall construct the nontrivial mCG by properly generalizing the model in [2, 3]. The starting point is the mapping of the dimensionally reduced Z_p model onto the mCG. The introduction of the θ-term allows for the mCG to have condensate dyonic phases. The fundamental issue of charge neutrality [4] will be recovered by means of an uniform neutralizing background charge density.

The original model is defined in terms of a two parameters lattice action:

$$S=\frac{1}{2g}\sum(\Delta_\mu\phi_j-S_{\mu j})^2-i\sum n_j\phi_j+i\frac{\theta}{4\pi}\epsilon_{\mu\nu}\epsilon_{ij}\sum(\Delta_\mu\phi_i-S_{\mu i})(\Delta_\nu\phi_j-S_{\nu j}) \quad (1)$$

ϕ_j, $j = 3, 4$, is a doublet of boson fields, n_j is the "electric charge density", the fields $S_{\mu j}$ are defined in terms of a constraint that relates them to the "magnetic charge distributions" m_j:

$$\Delta_x S_{y4} - \Delta_y S_{x4} = m_3 \quad ; \; -\Delta_x S_{y3} + \Delta_y S_{x3} = m_4 \tag{2}$$

The corresponding "Boltzmann weight" in the total partition function is a function of n_j and m_j and is given by:

$$\int \prod_j D\phi_j \int \prod_{\mu j} DS_{\mu j} \exp\left[-S\right] \delta[\Delta \times \vec{S}_4 - m_3] \delta[\Delta \times \vec{S}_3 + m_4] \tag{3}$$

Solving the constraint in eq.(3) one can either derive the "gauge representation" (**GR**) or the mCG representation of the CR-model.

In the GR one takes as dynamical variables the fields ϕ_j and the Lagrange multipliers, $\tilde{\phi}_j$, introduced in order to solve the constraint in eq.(3) according to the identity:

$$\delta(\Delta_x S_{y4} - \Delta_y S_{x4} - m_3) = \int D\tilde{\phi}_3 \exp[i \sum (\Delta_x S_{y4} - \Delta_y S_{x4} - m_3)]$$

and a similar one $(4 \leftrightarrow 3)$ for the other constraint.

Integrating out the fields $S_{\mu j}$ provides the action as a function of the fields ϕ_j and $\tilde{\phi}_j$. In the $\theta = 0$ case one obtains:

$$S = \frac{g}{2} \sum (\Delta_\mu \tilde{\phi}_j)^2 + i \sum (\vec{\Delta}\phi_4 \times \vec{\Delta}\tilde{\phi}_3 - \vec{\Delta}\phi_3 \times \vec{\Delta}\tilde{\phi}_4) + i \sum (n_j \phi_j + m_j \tilde{\phi}_j) \tag{4}$$

Such a representation is useful in order to trace out the realization of the mCG in terms of a 2D Conformal Field Theory as discussed, for example, in [5]. Here we want to focus our analysis on the IR properties of the system, which we shall analyze within the RG framework. The standard RG analysis is usually carried out within the mCG representation [4]. Then from now on we shall use such a representation throughout the rest of the paper. Here it is worth to mention that the gauge invariance in the original 4-dimensional gauge theory [2] has a remnant in the invariance of the action in eq.(4) under a constant shift of the fields ϕ_j, $\tilde{\phi}_j$:

$$\phi_j \to \phi_j + \alpha_j \;\; ; \;\; \tilde{\phi}_j \to \tilde{\phi}_j + \beta_j$$

The requirement of invariance under such a symmetry determines the charge neutrality constraints:

$$\sum n_j = \sum m_j = 0$$

We will study solutions consistent with these constraints, where an uniform background charge neutralizes the total charge of a condensate of particles, all with the same charge.

At this point we must remark that, in performing the summations in eq.(1), we have made no difference between the lattice and its dual. We are alleged to do so because we use the lattice only in order to regularize the gauge theory action. All the relevant RG analysis will be carried out within the framework of the continuum CG model [4] and any distinction between the lattice and its dual will be lost anyway.

Let us now work out the mCG formulation with a background charge. The θ-term in eq.(1) generates an additional "minimal coupling" that adds up to the "minimal electric coupling" term to give:

$$-i \sum \left(n_j + \frac{\theta}{2\pi} m_j \right) \phi_j \equiv -i \sum q_j \phi_j$$

Now we choose configurations where the charge densities are given by:

$$n_j(r) = \bar{n}_j + \nu_j(r) \quad ; \; m_j(r) = \bar{m}_j + \mu_j(r) \tag{5}$$

where $\bar{n}_j, \bar{m}_j$ are uniform solutions so that the neutrality conditions read as:

$$\bar{n}_j S + \sum_r \nu_j(r) \equiv \bar{N}_j + \sum_r \nu_j(r) = 0$$

$$\bar{m}_j S + \sum_r \mu_j(r) \equiv \bar{M}_j + \sum_r \mu_j(r) = 0$$

where S is the area of the sample.

According to the form of the charge density we have chosen, we can split $S_{\mu j}$ as $S_{\mu j} = \bar{S}_{\mu j} + \sigma_{\mu j}$, where:

$$\epsilon_{\mu\nu}\epsilon_{ij}\Delta_\mu \bar{S}_{\nu i} = m_j \quad ; \; \Delta_\mu \bar{S}_{\mu j} = 0$$

and

$$\epsilon_{\mu\nu}\epsilon_{ij}\Delta_\mu \sigma_{\nu i} = \mu_j \tag{6}$$

The solutions for $\bar{S}_{\mu j}$ are given by:

$$\bar{S}_{\mu j} = \epsilon_{\mu\nu}\Delta_\nu \epsilon_{ij}\Phi_j(r) \tag{7}$$

where:

$$\Phi_j(r) = \frac{\bar{m}_j}{4} r^2$$

Moreover the total density of charge q_j will be given by:

$$q_j = \bar{n}_j + \frac{\theta}{2\pi}\bar{m}_j + \nu_j + \frac{\theta}{2\pi}\mu_j \equiv \bar{q}_j + \epsilon_j \tag{8}$$

Then we split also ϕ_j as follows:

$$\phi_j = \bar{\phi}_j + \varphi_j \tag{9}$$

where

$$\frac{1}{g}(\Delta)^2\bar{\phi}_j + i\bar{n}_j = 0$$

The solutions for $\bar{\phi}_j$ are given by:

$$i\bar{\phi}_j = \frac{g\bar{n}_j}{4} r^2$$

After disregarding irrelevant constant terms, one obtains the total action in terms of the fluctuating fields and charge densities:

$$\tilde{S} = \frac{1}{2g}\sum(\Delta_\mu \varphi_j - \sigma_{\mu j})^2 - i\sum \epsilon_j \varphi_j + \sum[\Phi_j(r)\mu_j(r) + \bar{\phi}_j(r)\epsilon_j(r)] \tag{10}$$

Eq.(10) is almost the same result that one would obtain for a neutral charge distribution [2, 3]. The only difference is provided by the term:

$$\sum[\Phi_j(r)\mu_j(r) + \bar{\phi}_j(r)\epsilon_j(r)]$$

that describes the coupling of μ_j and ν_j to the background charge densities.

Integration over the fields φ_j is now straightforward. It yields:

$$\tilde{S} = -\sum_{r,r'}\left[\frac{\mu_3(r)\mu_3(r') + \mu_4(r)\mu_4(r')}{2g} + \frac{g}{2}[\epsilon_3(r)\epsilon_3(r') + \epsilon_4(r)\epsilon_4(r')]\right] G(r,r') +$$

$$\sum[\Phi_j(r)\mu_j(r) + \bar{\phi}_j(r)\epsilon_j(r)] - i\sum_{rr'}(\nu_3(r)\mu_4(r') - \nu_4(r)\mu_3(r'))\Theta(r-r') \tag{11}$$

$G(r,r^{'})$ is the longitudinal Green function of the fluctuating field φ_j. It must be evaluated with a proper regularization. In the continuum limit it is the usual Green function for a massless boson field in two dimensions:

$$G(r,r^{'}) \approx \ln\left|\frac{r-r^{'}}{a}\right|$$

where a is a length scale that works as an UV regularizator of the theory.

The "transverse" Green function $\Theta(r)$ is defined such that:

$$\partial_\mu \Theta(r) = \epsilon_{\mu\nu}\partial_\nu G(r,r^{'}) \tag{12}$$

that implies:

$$\Theta(r) = \arctan\left(\frac{y}{x}\right) \tag{13}$$

An important remark concerns the consequences of the Bohm-Aharonov (**BA**) term that appears in eq.(11). It makes the partition function ill defined, unless the combinations $\nu_i^3\mu_j^4 - \nu_j^4\mu_i^3$ are integer numbers. From now on we will make the assumption that $\nu_i^{3,4}$ and $\mu_i^{3,4}$ are integers [4].

Eq.(11) leads us to the first relevant result of our work. We have written down a model for interacting particles with both electric and magnetic charge. Each dyon takes either a type-3 or a type-4 charge. Then we define the nonuniform part of the charge densities as a set of N particles carrying either type-3 or type-4 charges:

$$\mu^j(r) = \sum_{k=1}^{N}\mu_k^j\delta(r-r_k) \quad ; \; \nu^j(r) = \sum_{k=1}^{N}\nu_k^j\delta(r-r_k)\ , j=3,4 \tag{14}$$

At fixed charge distribution the action will be given by [5]:

$$S[\{\nu^j\},\{\mu^j\}] = \sum_{i\neq k=1}^{N}\frac{1}{2}\left[\frac{\mu_i^3\mu_k^3+\mu_i^4\mu_k^4}{g} + g(\epsilon_i^3\epsilon_k^3+\epsilon_i^4\epsilon_k^4)\right]\ln\left|\frac{r_i-r_k}{a}\right|$$

$$- i\sum_{i\neq k}(\nu_i^3\mu_k^4 - \nu_k^4\mu_i^3)\phi(r_i-r_k) - \frac{1}{4}\sum_i\left(g(\bar{q}^3\epsilon_i^3+\bar{q}^4\epsilon_i^4) + \frac{\bar{m}\mu_i^3+\bar{m}\mu_i^4}{g}\right)r_i^2 \tag{15}$$

In order to perform a RG analysis of the mCG we need the grand canonical partition function, Z_{GC}. From now on we will neglect the index j (we shall later justify such an assumption). Z_{GC} is written in terms of the "fugacities" $Y(\nu,\mu)$, given by the probability of creating a particle with charges ν,μ at fixed values of

the parameters. In the continuum formalism the mCG partition function is given by:

$$Z_{GC} = \sum_{\{\nu,\mu\}} = \int \prod_j \left[Y(\nu_j, \mu_j) \frac{d^2 r_j}{a^2} \right] e^{-S[\{\nu,\mu\}]} \tag{16}$$

3 Linear RG analysis.

In this section we shall derive the RG equations at first order in the fugacities and will use them in order to characterize the condensate phases. The widely used RG analysis can be improved by adding higher order (nonlinear) corrections in Y.

The (first-order) RG equations are derived by rescaling the cutoff, $a \to a + da$, and then by "reabsorbing" the corresponding changes in the partition function by means of a redefinition of the fugacities [4].

If $a \to a + da$, one has:

$$\int \prod_j \left[Y(\nu_j, \mu_j) \frac{d^2 r_j}{a^2} \right] e^{-S[\{\nu,\mu\}]} \to$$

$$\int \prod_{j=1}^{N} \left\{ \left[Y(\nu_j, \mu_j) \frac{d^2 r_j}{a^2} \right] \left[1 - \frac{da}{a} \left(2 + \frac{1}{2g} \mu_j (\bar{M} - \mu_j) + \frac{g}{2} \epsilon_j (\bar{Q} - \epsilon_j) \right) \right] \right\} e^{-S[\{\nu,\mu\}]} + O(\left(\frac{da}{a}\right)^2) \tag{17}$$

In eq.(17) $Y(\nu_j, \mu_j)$ have to be understood as "running" fugacities, whose dependence on a is fixed by the requirement that Z_{CG} <u>does not depend on a</u>. Accordingly, one gets the following renormalization of the fugacities:

$$Y(\nu_j, \mu_j) \to Y(\nu_j, \mu_j) + dY(\nu_j, \mu_j) \tag{18}$$

with

$$a \frac{dY(\nu_j, \mu_j)}{da} = x(\nu_j, \mu_j) Y(\nu_j, \mu_j)$$

$$x(\nu_j, \mu_j) = 2 + \frac{1}{2g} \mu_j (\bar{M} - \mu_j) + \frac{g}{2} \epsilon_j (\bar{Q} - \epsilon_j) \tag{19}$$

Following [4] we assume that the set of particles that condense is defined by the values of the charges that maximize each $x(\nu_j, \mu_j)$, constrained by the global neutrality conditions:

$$\sum_j \nu_j = -\bar{N} \quad ; \; \sum_j \mu_j = -\bar{M}$$

The solution is given by:

$$\nu_1 = \ldots = \nu_N \equiv \bar{\nu} = -\frac{\bar{N}}{N} \quad ; \; \mu_1 = \ldots = \mu_N \equiv \bar{\mu} = -\frac{\bar{M}}{N} \tag{20}$$

and the corresponding value of the exponent is:

$$x(\bar{\nu}, \bar{\mu}) = 2 + \frac{1}{2g}\bar{\mu}(\bar{M} - \bar{\mu}) + \frac{g}{2}\bar{q}(\bar{Q} - \bar{q}) \approx 2 + \frac{1}{2g}\bar{\mu}\bar{M} + \frac{g}{2}\bar{q}\bar{Q} \tag{21}$$

where we have introduced the "generalized charges" $\bar{q} = \bar{\nu} + \frac{\theta}{2\pi}\bar{\mu}$.

The outcome of the linearized RG analysis in the presence of a background is that the condensate phases are given by a gas of particles carrying both electric and magnetic charges (**dyons**) of the same sign and all identical, corresponding to a breaking of both P and C. Being all the charges equal, the BA term disappears and $S[\{\mu^j\}, \{\nu^j\}]$ becomes the sum of two independent contributions. This justifies our neglecting the index j and the BA-term in the RG analysis.

The breaking of the discrete symmetries P and C implies that the underlying field theory breaks them as well, and we believe it to be related to what happens in a chiral 2D CFT [8].

The partition function for the condensate of N dyons with charges $\bar{\mu}$, $\bar{\nu}$ is given by:

$$Z^*_{\bar{\mu}\bar{\nu}} = \int \frac{d^2 r_1}{a^2} \ldots \frac{d^2 r_N}{a^2} \exp\left[\left(\frac{1}{2g}\bar{\mu}^2 + \frac{g}{2}\bar{\epsilon}^2\right)\sum_{i \neq j} \ln\left|\frac{r_i - r_j}{a}\right| - \frac{1}{4}\left(\frac{\bar{\mu}\bar{m}}{g} + g\bar{q}\bar{\epsilon}\right)\sum_j r_j^2\right] \tag{22}$$

4 Nonlinear RG approximation.

In this section we shall extend the analysis of section[3] by including higher order processes that eventually lead to a renormalization of the interaction strength between dyons.

In order to write down higher order RG equations we will perturb around the condensate with a fluctuation given by a particle/antiparticle pair. This is just a different way of describing the "charge annihilation" process analyzed in [4]. The action for the condensate plus the pair is given by:

$$S = S_N + \Delta S$$

where S_N is the action for the unperturbed condensate and:

$$\Delta S = \left(\frac{\mu_P^2}{2g} + \frac{g}{2}\epsilon_P^2\right) \ln\left|\frac{R_+ - R_-}{a}\right| - \sum_{j=1}^{N} \left(\frac{\mu_P \bar{\mu}}{2g} + \frac{g}{2}\epsilon_P \bar{q}\right) \left\{\ln\left|\frac{R_+ - r_j}{a}\right| - \ln\left|\frac{R_- - r_j}{a}\right|\right\} \tag{23}$$

Notice that we have neglected the BA-term; infact we shall prove that μ_P, ν_P must be equal to $\bar{\mu}, \bar{\nu}$, which justifies that. R_+ and R_- are the coordinates of the particle and of the antiparticle, respectively, $\pm\mu_P$ and $\pm\epsilon_P$ are the charges of the two particles in the pair.

In order to find the correction to the coupling constant, we must sum over all configurations with $a < |R_+ - R_-| < a + da$. Following [4], we define:

$$R_+ = R + \frac{s}{2} \quad ; \; R_- = R - \frac{s}{2}$$

At second order in s we have:

$$\Delta S = \left(\frac{\mu_P^2}{2g} + \frac{g}{2}\epsilon_P^2\right) \ln\left|\frac{s}{a}\right| + \frac{1}{2}\left[\frac{\mu_P \bar{m}}{g} + \frac{g}{2}\epsilon_P \bar{n}\right] s \cdot R - \sum_j \left(\frac{\bar{\mu}\mu_P}{2g} + \frac{g}{2}\bar{q}\epsilon_P\right) s \cdot \frac{(R - r_j)}{|R - r_j|^2} \tag{24}$$

Summing over the relevant range of values of s, we get the following contribution to the N-particle condensate partition function:

$$\int \frac{d^2R}{a^2} \int_{a<|s|<a+da} \frac{d^2s}{a^2} |s|^{-\left(\frac{\mu_P^2}{2g} + \frac{g}{2}\epsilon_P^2\right)} \exp\left\{s \cdot \frac{\partial}{\partial R}\psi(R)\right\} Y(\mu_P, \nu_P) Y(-\mu_P, -\nu_P) \tag{25}$$

where the function ψ is given by:

$$\psi(R) = -\left(\frac{\bar{\mu}\mu_P}{2g} + \frac{g}{2}\bar{q}\epsilon_P\right) \sum_j \ln\left|\frac{R - r_j}{a}\right| + \frac{1}{4}\left[\frac{\mu_P \bar{m}}{2g} + \frac{g}{2}\epsilon_P \bar{q}\right] R^2$$

Let us define $y^2_{\mu_P,\nu_P} \equiv Y(\mu_P, \nu) Y(-\mu_P, -\nu_P)$. The linear RG equation for $y^2_{\nu_P,\nu_P}$ is:

$$\frac{dy^2_{\mu_P,\nu_P}}{d\ln a} = 2\left[2 - \frac{\mu_P^2}{2g} - \frac{g}{2}\nu_P^2\right] y^2_{\mu_P,\nu_P} \tag{26}$$

Then the operator y_{μ_P,ν_P} is relevant for

$$2-\frac{\mu_P^2}{2g}-\frac{g}{2}\left(\nu_P+\frac{\theta}{\pi}\mu_P\right)>0 \tag{27}$$

Eq.(27), together with the $SL(2,Z)$-symmetry of the model (see next section), allows us to determine the global phase diagram. For example we shall see that shifting $\frac{\theta}{2\pi}$ by -1 is equivalent to shifting ν_P to $\nu_P+\mu_P$ and $\bar{N}$ to $\bar{N}+\bar{M}$. The important consequence of the discrete symmetry is the map between regions of parameter space corresponding to different condensate phases, one onto the other. This generates the global phase diagram that is quite similar to the one derived in [3] (see Fig.1). However the background does definitely modify the RG flow, as we will see later.

Eq.(25) may be expanded at second order in s. The result is:

$$\text{const}-\frac{\pi}{2}y^2\frac{da}{a}\int d^2R\psi(r)\nabla^2\psi(r)=$$

$$\text{const}-\frac{\pi}{2}y^2\frac{da}{a}\left(\frac{\mu_P\bar{\mu}}{2g}+\frac{g}{2}\epsilon_P\bar{q}\right)\sum_{i\neq j=1}^{N}\ln\left|\frac{r_i-r_j}{a}\right|+\frac{\pi}{2}y^2\frac{1}{2}\frac{da}{a}\left(\frac{\mu_P\bar{m}}{2g}+\frac{g}{2}\nu_P\bar{n}\right)\sum_{j=1}^{N}r_j^2 \tag{28}$$

Eq.(28) corresponds to a renormalization in the coupling constant:

$$\alpha^2_{\bar{\mu}\bar{\nu}}=\frac{\bar{\mu}^2}{g^2}+g\bar{q}^2$$

In fact, by summing over all the possible fluctuations, one obtains:

$$d\alpha^2_{\bar{\mu}\bar{\nu}}=-\sum_{\nu_P,\mu_P}\frac{\pi}{2}y^2_{\mu_P,\nu_P}\frac{da}{a}\left(\frac{\mu_P\bar{\mu}}{2g}+\frac{g}{2}\epsilon_P\bar{q}\right)^2 \tag{29}$$

With our choice of the parameters only one of the $y^2_{\mu_P\nu_P}$ is relevant. Consistency with the linear RG analysis shows that the relevant one is $y_{\bar{\mu}\bar{\nu}}$. Then we can approximate the right hand side of eq.(28) as:

$$\frac{d\alpha_{\bar{\mu}\bar{\nu}}}{d\ln a}=-\frac{\pi}{2}y^2_{\bar{\mu}\bar{\nu}}(\alpha_{\bar{\mu}\bar{\nu}})^2 \tag{30}$$

The solution of eq.(30)is given by:

$$\frac{1}{\alpha_{\bar{\mu},\bar{\nu}}(a)}-\frac{1}{\alpha_{\bar{\mu},\bar{\nu}}(a_0)}=\frac{\pi}{2}\int_{a_0}^{a}\frac{da}{a}y^2_{\bar{\mu},\bar{\nu}} \tag{31}$$

At very large scales we find the values of the interaction strength corresponding to the IR fixed point of the system. The fixed point strength is given by:

$$\lim_{a\to\infty} \alpha_{\bar{\mu},\bar{\nu}}(a) = 0 \tag{32}$$

Since $\alpha_{\bar{\mu},\bar{\nu}}$ is a nonnegative definite function, eq.(32) is satisfied if, and only if:

$$\lim_{a\to\infty} \frac{1}{g(a)} = 0 \quad ; \; \lim_{a\to\infty} \frac{\theta}{2\pi}(a) = -\frac{\bar{\nu}}{\bar{\mu}} \tag{33}$$

Eq.(33) tells us what the IR fixed points of the model are in the plane ($\frac{\theta}{2\pi}$, $\frac{1}{g}$), precisely they all lie on the real axis at $\frac{\theta}{2\pi}$ equal to any rational number. Such a result is deeply related to the properties of the group $SL(2, Z)$. Infact all the RG fixed points of the mCG are $SL(2, Z)$-fixed points too. Then it is possible to generate the manifold of all these points by letting $SL(2, Z)$ act on anyone of them, and consequently the local RG flow is mapped also onto the flow around any other fixed point.

5 Discrete symmetries and global phase diagram.

In the previous sections we have employed RG equations in order to find informations about the condensate phases of the system. The RG approach only provides local informations on the phase diagram. Instead its global properties are a consequence of the discrete group $SL(2, Z)$(the modular group), which is a symmetry of the model.

Introducing the complex variable $\zeta = \frac{\theta}{2\pi} + i\frac{1}{g}$, the group $SL(2, Z)$ acts as:

$$\zeta \rightarrow \frac{A\zeta + B}{C\zeta + D} \tag{34}$$

where:

$$M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a matrix whose entries are relative integers and whose determinant is one. It is well known [9] that the full group is generated by repeated application of the following two transformations:

- **Translations**

$$T \quad : \zeta \rightarrow \zeta + 1;$$

- **Inversions**

$$S \ : \zeta \to -1/\zeta$$

The action of $SL(2,Z)$ on the mCG is defined both on the parameter space and on the charge distributions as follows:

$$\zeta \to \zeta + 1 \ ; \ \mu \to \mu \ ; \ \nu \to \nu + \mu \ ; \ \bar{M} \to \bar{M} \ ; \ \bar{N} \to \bar{N} + \bar{M}$$

$$\zeta \to -\frac{1}{\zeta} \ ; \ \mu \to -\nu \ ; \ \nu \to \mu \ ; \ \bar{M} \to -\bar{N} \ ; \ \bar{N} \to \bar{M} \tag{35}$$

i.e., they act simultaneously on the complex parameter space and on the background, which fixes also the dyon charges.

Nonlinear RG analysis provides us with the relevance condition for $y^2_{\bar{\mu}\bar{\nu}}$, eq.(27). Such a condition determines a domain in the $\left(\frac{\theta}{2\pi}, \frac{1}{g}\right)$-plane we shall refer to as the "relevance domain" for the dyon condensate of charges $\bar{\mu}, \bar{\nu}$. It is appropriate to notice that the dyonic phases with $\frac{\theta}{2\pi} \neq 0$ realize in 2D the t'Hooft confined phases [1].

Using the full $SL(2,Z)$-symmetry, as defined in eq.(35), one can generate the regions of attraction for all possible condensates, which do not overlap. The phase boundaries are "semicircle" mapped one onto the other by $SL(2,Z)$-transformations. Therefore we have uniquely determined the phase diagram and its boundaries, which cover all the upper right complex plane (see Fig.1). We should notice that the modular group should also help in deriving the full beta-function in whole the upper complex plane in terms of modular forms (see [15] for an attempt).

6 The critical theory for the fixed points: relation to FQHE.

In this section we make clear our main conjecture about the field theory description of the attractive fixed points.

To derive the critical theory from the general CG model is a difficult task. Still one can try to guess it and check its consistency with the general description in terms of a mCG. In the following we propose such a field theory, at the IR fixed points, as the 2D CFT description [8] of the QH condensates at Laughlin's fillings and can eventually be generalized at least for Jain's fillings [14].

As it has been seen, all the phases break P and C and they correspond to a Coulomb Gas with an uniform classical background. The regions of attraction

are determined by the value of the background charge densities and are related one to the other by $SL(2,Z)$-transformations. This determines a "hierarchy", so that the relevance domains"shrink" as one goes deeper into the hierarchy level (see Fig.1). Finally the charge ratios for the condensate are in agreement with the interpretation in terms of quasiparticles with fractional charge [13]. Then, we hypothesize that, at the IR stable fixed points, the correct field theory limit of our model is given by a chiral CFT with central charge $c = 1$, described in terms of a compactified scalar field with radius given by $R^2 = 2p+1$. In such a theory, the conformal blocks correspond to the vertex operators:

$$: e^{i\alpha_l \phi(z)} : \ ; \ \alpha_l = \frac{l}{R}; l = 1 \dots 2p+1$$

The momentum lattice $\{p/R\}$ and the weight lattice $\{R^2 w\}$ describe the eletric and the magnetic charge of "anyons", respectively. When $l = 2p+1$ we get the "electron" operator. Such a theory has been successfully employed in the description of the FQHE plateaux at filling $\nu = 1/(2p+1)$ [8]. In conclusion it seems appropriate to us to identify the field-theoretical description of the IR fixed point with $\frac{\theta}{2\pi} = -\frac{\bar{\nu}}{\bar{\mu}}$ with one of the CFT described above at the proper value of R.

By acting with $SL(2,Z)$ on a single IR fixed point one generates all the others, thus strengthening the idea of an unified description of all the IR fixed points (and the Hall plateaus we identify with them). Previous work on this subject [5, 12] seems to strongly support such an identification, too.

On the other hand the mGG, here presented, is directly connected with Laughlin plasma description [13]. Infact it is well kwown that the square modulus of the ground state wavefunction of a QH-condensate at filling $1/(2p+1)$ is given by:

$$|\Psi(z_1, \dots, z_{N_e})|^2 = \prod_{i<j=1}^{N_e} |z_i - z_j|^{2(2p+1)} e^{-\frac{1}{2}\sum_{j=1}^{N_e} |z_j|^2} \tag{36}$$

It is evident that eq.(36) is exactly the partition function in eq.(22) for a dyonic condensate whose electric-to-magnetic charge ratio, given by $-\frac{\theta_*}{2\pi}$, is equal to $1/(2p+1)$.

So we can be confident on the correctness of such a guess, which should be thoroughly studied.

7 Conclusion and perspectives.

In this paper we have discussed a modified version of the CR mCG. The presence of the background term in such a generalized CG model allows for some new properties:

1. Getting non-neutral condensate phases, related by $SL(2,Z)$ as its attractive fixed points. They are rational points on the $\frac{\theta}{2\pi}$-axis (see Fig.1). The related condensate is made out of dyons with electric charge $\bar{\nu}$ and magnetic charge $\bar{\mu}$, such that $\frac{\theta_*}{2\pi} = -\frac{\bar{\nu}}{\bar{\mu}}$.
2. Driving the RG flow of the system by means of tunable external parameters, given by the electric (magnetic) background densities $\bar{N}(\bar{M})$;
3. Defining the global phase diagram by means of the $SL(2,Z)$-symmetry;

A possible field theory for the attractive fixed points has been proposed as a chiral CFT, at least for the simple case $\frac{\theta_*}{2\pi} = \frac{1}{2p+1}$. Then the mCG seems to be consistent with the physics of the FQHE. In this framework the identification of the parameters $\frac{1}{g}$ and $\frac{\theta}{2\pi}$ as the longitudinal and the Hall conductance, respectively, appears natural. However the denominator of $\frac{\theta_*}{2\pi}$, $\bar{\mu}$, is not constrained to be odd, as it should be. This is related to the wrong choice of the discrete symmetry group, the right one being the group $\Gamma(2)$ discussed in [10, 11]. Anyway the relation of the present model with the FQHE physics deserves further work. We look forward to completing the analysis of the global phase diagram in particular of the unstable fixed points, which describe the transition between two different fillings. The modular invariance should imply, in our opinion, the (super) universality of the transition, advocated by experimental [16] and theoretical work [laws of corresponding states [17]]. As a final remark we point out the possible role of such a model in the study of non-commutative gauge theories.

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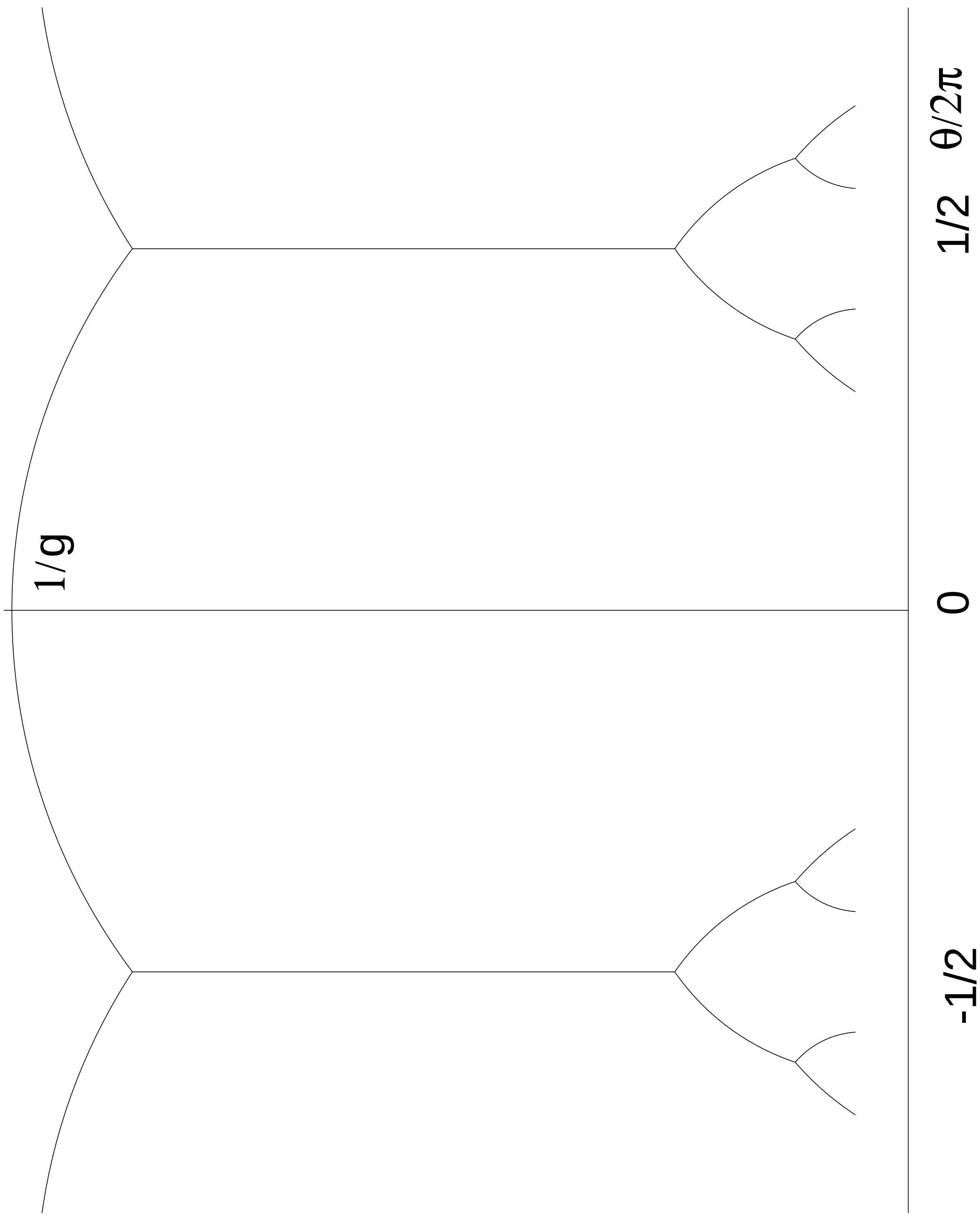

Figure 1: The global structure of the phase diagram